November 18th, 2005

Non-Geometric Magnetic Flux and Crossed Modules

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Abstract

It is shown that the BRST operator of twisted $N = 4$ Yang-Mills theory in four dimensions is locally the same as the BRST operator of a fully decomposed non-Abelian gerbe. Using locally defined Yang-Mills theories we describe non-perturbative backgrounds that carry a novel magnetic flux. Given by elements of the crossed module $G \ltimes \mathrm{Aut}\, G$, these non-geometric fluxes can be classified in terms of the cohomology class of the underlying non-Abelian gerbe, and generalise the centre $\mathrm{Z}\, G$ valued magnetic flux found by 't Hooft. These results shed light also on the description of non-local dynamics of the chiral five-brane in terms of non-Abelian gerbes.

Keywords Non-Abelian Gerbes, twisted Yang-Mills, M-Theory, Five-branes.
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Contents

1 Introduction

Among all the four-dimensional interacting quantum field theories the supersymmetric Yang-Mills theory is perhaps the best understood. It enjoys beneficial symmetries that eliminate infinities in perturbation theory on two different levels — first, as a gauge theory and, second, as a maximally supersymmetric quantum field theory. In addition to this it turns out that the theory is conformal, the beta-function vanishes, and that it enjoys an exact non-perturbative symmetry, S-duality.

The underlying mathematical structure to gauge theory on a general manifold X is that of a principal G-bundle: fields on overlapping neighbourhoods $\mathcal{U}_i$ and $\mathcal{U}_j \subset X$ can differ by a gauge transformation $g_{ij} \in G$ on the overlap $\mathcal{U}_i \cap \mathcal{U}_j = \mathcal{U}_{ij}$ in a consistent way. Consistency here means that passing through a third neighbourhood we get back to where we started $g_{ij}g_{jk}g_{ki} = \mathbf{1}$. The traditional way to describe a physical system on X is indeed in terms of locally, say on $\mathcal{U}_i$, defined differential equations. Sometimes this local quality of differential equations in Physics can be misleading, as some of the local fields should more properly be accommodated to intersections $\mathcal{U}_{ij}$ rather than on $\mathcal{U}_i$. Yet there is nothing in the differential equations in themselves to give away this difference in character. This phenomenon occurred for instance in [1] where the Stückelberg field associated to a two-form turned out to be the connection one-form of an only locally defined line bundle in the structure of an underlying Abelian gerbe.

In this paper we investigate $N = 4$ supersymmetric Yang-Mills theory where the consistency condition $g_{ij}g_{jk}g_{ki} = \mathbf{1}$ has been relaxed, though in a controlled way. On a non-Abelian gerbe we may indeed allow for such "inconsistencies" in the way in which the global structure of the theory is put together from local pieces. We investigate in particular configurations where $N = 4$ supersymmetric Yang-Mills theory is localised on double intersections $\mathcal{U}_{ij}$, and in a generic local neighbourhood $\mathcal{U}_i$ the theory is a slightly truncated version thereof. From the outset there is no reason to wish to write down such configurations; this is, however, what emerges by studying how the symmetries on the twisted Yang-Mills theory can be embedded in the global structure of a non-Abelian gerbe.

The underlying technical reason that allows us to make use of non-Abelian gerbes in $N = 4$ supersymmetric Yang-Mills theory is indeed the observation that the BRST symmetry of a general non-Abelian gerbe [2] is, with certain qualifications, the same as the BRST symmetry of the twisted $N = 4$ supersymmetric Yang-Mills theory [3–5]. The main novelty is that the global structure of the non-Abelian gerbe is loose enough to include non-perturbative symmetries of the quantum theory. This makes it possible to describe new non-geometric super-Yang-Mills backgrounds in field theory, where local fields in overlapping neighbourhoods are related to each other by an S-duality transformation. In [6] the non-perturbative symmetry was T-duality, hence the term "non-geometric".

There are at least two ways to interpret the new structure on the overlaps $\mathcal{U}_{ij}$. The most straightforward is perhaps to think of this as a twisting of the global fields on X by some local extra structure. This is the rôle played by the connective structure of an Abelian gerbe in Hitchin's generalised geometry, for

instance. The other approach is to interpret the new structure as dynamical degrees of freedom localised in certain parts of the space-time X. Perhaps a more familiar example of similar behaviour is the fact that the presence of branes or other defects introduces degrees of freedom on the worldvolume of these objects [7–9].

Whichever point of view one wishes to take, the new structure will give rise to non-geometric magnetic fluxes in terms of the topological class of the gerbe. These fluxes are generalisations of the magnetic flux found by 't Hooft by studying loop operators in gauge theory [10,11]. In a certain sense the novel fluxes can be though of as non-Abelian surface holonomies, analogously to as how 't Hooft's magnetic fluxes arise from holonomies over closed loops.

The Paper is organised as follows: In Sec. 2 we recall aspects of super-Yang-Mills and the twisting procedure. In Sec. 3 we quote the BRST symmetry of the non-Abelian gerbe from [2], and show how it reduces to the BRST symmetry of the Yang-Mills theory. In doing so it is important to notice that this matching is functionally different on local charts $\mathcal{U}_i$ and $\mathcal{U}_j$ from the matching on double intersections $\mathcal{U}_{ij}$. In Sec. 4 we consider a non-geometric example where the local description of the Yang-Mills theory on adjacent charts is related by S-duality. In Sec. 5 we generalise 't Hooft's magnetic flux to the non-geometric magnetic flux of a non-Abelian gerbe that takes its values in the crossed module associated to the gerbe. Finally, in Sec. 6 we comment on what implications the present results have for modelling the local dynamics of chiral five-branes in terms of non-Abelian gerbes.

2 The $N = 4$ supersymmetric Yang-Mills theory

In this section some of the basics of the $N = 4$ supersymmetric Yang-Mills theory as a quantum theory are reviewed, including electric-magnetic duality, and twists to topological theories.

The fields in the $N = 4$ supersymmetric Yang-Mills theory based on the Lie-group G belong all to the adjoint representation of the Lie-algebra $\mathrm{Lie}\, G$. Apart from the local gauge symmetry, also the global automorphisms $\mathrm{SU}(4)_\mathrm{R}$ of the $N = 4$ supersymmetry algebra act on these fields. The field content of the theory is as follows:

– Local gauge field A;

– Gaugino λ, $(\bar{\lambda})$ in the fundamental representation $\mathbf{4}$ (resp. $\bar{\mathbf{4}}$) of $\mathrm{SU}(4)_\mathrm{R}$;

– Scalars Φ in the antisymmetric representation $\mathbf{6}$ of $\mathrm{SU}(4)_\mathrm{R}$.

The coupling constant and the θ-angle fit into the complex combination

$$\tau \quad = \quad \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \; . \tag{1}$$

In addition to the local gauge symmetry and the R-symmetry group $\mathrm{SU}(4)_\mathrm{R}$, the quantum theory is invariant under the S-duality group $\mathrm{SL}(2,\mathbb{Z})$ (for simply laced gauge groups) generated by

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ . \tag{2}$$

The S-duality group acts on τ by fractional linear transformation.

The S-transformation of the S-duality group generates electric-magnetic duality transformations where the electric field strength $F = \mathrm{d}\, A + A \wedge A$ is replaced by its Hodge dual $\tilde{F} = \star F$. In general there is no guarantee for the existence of a corresponding magnetic gauge field $\tilde{A}$, and this relationship holds indeed only using equations of motion and in a suitably fixed gauge. This duality transformation changes also the electric gauge group G itself to its magnetic dual G^v. The global structure of the quantum theory should, therefore, involve both gauge groups, G and G^v [12]. A more precise statement demonstrated in [13] is that the Wilson-'t Hooft operators can be labelled by elements of the electric and magnetic weight-lattices $(\Lambda_w \oplus \Lambda_{mw})/\mathcal{W}$ modulo the action of the common Weyl group $\mathcal{W}$.

Twisting in $N = 2$ supersymmetric Yang-Mills was introduced in [14]. There, twisting means identifying the R-symmetry group $\mathrm{SU}(2)_\mathrm{R}$ with one of the factors of the Euclidean spin-group $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$. Generalisations to $N = 4$ were proposed in [3, 15]. In these cases twisting amounts to breaking the R-symmetry group $\mathrm{SU}(4)_\mathrm{R}$ to an $\mathrm{SU}(2)_\mathrm{R}$, and then proceeding as above. One way to distinguish twists is to determine how the fundamental $\mathbf{4}$ decomposes into representations of the four-dimensional spin-group. Up to interchanging left and right, there are three possibilities [4]:

Chiral twist:	$\mathbf{4} \longrightarrow (\mathbf{2},\mathbf{1}) \oplus (\mathbf{2},\mathbf{1})$
Half twist:	$\mathbf{4} \longrightarrow (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1})$
Non-chiral twist:	$\mathbf{4} \longrightarrow (\mathbf{2},\mathbf{2})$

In this paper we shall concentrate on the chiral twist that leads to the Vafa-Witten theory discussed in [4, 5]. This twist has a residual global symmetry $\mathrm{SU}(2)_F$ that interchanges the two copies of $(\mathbf{2},\mathbf{1})$. In twisted Yang-Mills theory this symmetry is explicitly broken by assigning different ghost numbers to members of the same multiplet.

Re-identification of the spin-group in the quantum theory changes the energy-momentum tensor and therefore, potentially, the underlying quantum theory itself. In a flat or hyper-Kähler background metric the twisted and the physical $N = 4$ super-Yang-Mills theories are nevertheless equivalent [4, 16]. Examples of such four manifolds are the compact K3, and the noncompact hyper-Kähler resolutions of orbifolds of the form $M = \mathbb{C}^2/\Gamma$, where $\Gamma \subset \mathrm{SU}(2)$ is a discrete subgroup and $\mathbb{C}^2$ its linear representation.

Action of twisting on fields			Ghost number		
SYM		*TYM*	*in TYM*	*in gerbe*	
A	$\longrightarrow$	A	0	0	
Φ	$\longrightarrow$	$B^{[2+]} \oplus \phi^{[0]} \oplus C^{[0]} \oplus \bar{\phi}^{[0]}$	$(0,2,0,-2)$	$(0,2,2,2)$	(3)
λ	$\longrightarrow$	$\chi^{[2+]} \oplus \tilde{\psi}^{[1]} \oplus \eta^{[0]} \oplus \zeta^{[0]}$	$(-1,1,-1,1)$	$(1,1,3,3)$	
$\bar{\lambda}$	$\longrightarrow$	$\psi^{[2+]} \oplus \tilde{\chi}^{[1]}$	$(1,-1)$	$(1,1)$	
$*$	$\longrightarrow$	$H^{[2+]} \oplus \tilde{H}^{[1]}$	$(0,0)$	$(2,2)$	

Table 1: *Field content of super-Yang-Mills (SYM) and its decomposition in the twisted theory, Topological Yang-Mills (TYM); Ghost numbers in TYM and the gerbe. Square brackets refer to the degree a differential form, and the superscript* [2+] *to a self-dual two-form.*

3 Locally twisted Yang-Mills on a gerbe

The topology of a non-Abelian gerbe can be given in terms of the cocycle data (g_{ijk}, λ_{ij}). Here λ_{ij} is an $\mathrm{Aut}\, G$ valued function on the double intersection of local charts $\mathcal{U}_{ij}$, and g_{ijk} is a G-valued function on the triple intersection of local charts $\mathcal{U}_{ijk}$. The distinction between G and $\mathrm{Aut}\, G$-valued objects is quite important. (This structure will be discussed in more detail in Sec. 5.3.) In general an automorphism can involve an outer part that cannot be effected by conjugation with a group element. In the case of a Lie-group, such outer automorphisms are symmetries of the Dynkin diagram. For instance, $\mathrm{Out}\,\mathrm{SU}(n) = \mathbb{Z}_2$ (complex conjugation, $n > 2$) and $\mathrm{Out}\,\mathrm{Spin}(8) = \mathbb{Z}_3$ (triality).

A fully decomposed gerbe was described in terms of the above cocycle in [17], and in terms of differential geometry in [18]. When the decomposition is only partial, one is lead to intermediate structures that involve local non-Abelian bundles but whose characteristic classes are Abelian in the sense of [17]. This is true of bundle gerbes [19].

The BRST operator that generates infinitesimal symmetries of a gerbe was constructed in [2]. The local fields involve the local connection which is a $\mathrm{Lie}\,\mathrm{Aut}\, G$ valued one-form m_i; a $\mathrm{Lie}\, G$ valued one-form γ_{ij} on the double intersection of local charts $\mathcal{U}_{ij}$; and a local $\mathrm{Lie}\, G$ valued two-form B_i. In Refs. [2,18] these fields are really *group* valued differential forms [20], though for the present discussion they reduce to algebra valued forms. To write the BRST operator down, we need the following notation:

- The adjoint action of the group element is denoted by $\iota_g(h) = ghg^{-1}$.
- Given an Hodge star $\star$, we denote $\iota^+_x = \frac{1}{2}(1+\star)\iota_x$ for any Lie-algebra valued two-form x.

	ghost#	0-form	1-form	2-form	3-form
G	0	g_{ijk}	γ_{ij}	$B_i,\ \delta_{ij}$	ω_i
	1	a_{ij}	$E_i,\ \eta_{ij}$	α_i	
	2	$\phi_i,\ b_{ij}$	ρ_i		
	3	σ_i			
$\mathrm{Aut}(G)$	0	λ_{ij}	m_i	ν_i	
	1	c_i	π_i		
	2	φ_i			

Table 2: *Fields and field strengths on the universal gerbe.*

– The covariant exterior derivative of Lie-algebra valued forms is

$$\mathrm{d}_{m_i} x \quad := \quad \mathrm{d}\, x + [m_i, x]\ . \tag{4}$$

– The action of an automorphism λ on an automorphism valued form m is denoted ${}^{\lambda}m$. For connection one-form we write ${}^{\lambda *}m$.

– The local field strength of m_i is

$$\kappa(m_i) \quad = \quad \mathrm{d}\, m_i + \frac{1}{2}[m_i, m_i]\ . \tag{5}$$

The fields appearing in a fully decomposed non-Abelian [18] gerbe and the associated BRST operator q are summarised in Table 2. The BRST operator of a fully decomposed gerbe [2] is

$$\begin{aligned}
\mathrm{q}m_i &= \pi_i + \iota_{E_i} - \mathrm{d}_{m_i} c_i && (6)\\
\mathrm{q}_c\gamma_{ij} &= \eta_{ij} + E_i - \lambda_{ij}(E_j) + \mathrm{d}_{m_i} a_{ij} - [\gamma_{ij}, a_{ij}] && (7)\\
\mathrm{q}_c B_i &= \alpha_i + \mathrm{d}_{m_i} E_i && (8)\\
\mathrm{q}_c\pi_i &= \iota_{\rho_i} + \mathrm{d}_{m_i}\varphi_i && (9)\\
\mathrm{q}_c E_i &= -\rho_i + \mathrm{d}_{m_i}\phi_i && (10)\\
\mathrm{q}c_i &= \varphi_i + \iota_{\phi_i} + \tfrac{1}{2}[c_i, c_i] && (11)\\
\mathrm{q}_c\eta_{ij} &= -\mathrm{d}_{m_i} b_{ij} + \rho_i - \lambda_{ij}(\rho_j) + [\iota_{\eta_{ij}} - \pi_i, a_{ij}] - [\varphi_i + \iota_{b_{ij}}, \gamma_{ij}] && (12)\\
\mathrm{q}_c\alpha_i &= \mathrm{d}_{m_i}\rho_i - [\nu_i, \phi_i] - [\pi_i, E_i] - [\varphi_i, B_i] && (13)\\
\mathrm{q}_c\varphi_i &= -\iota_{\sigma_i} && (14)\\
\mathrm{q}_c\phi_i &= \sigma_i && (15)\\
\mathrm{q}_c\rho_i &= \mathrm{d}_{m_i}\sigma_i + [\pi_i, \phi_i] + [\varphi_i, E_i] && (16)\\
\mathrm{q}_c\sigma_i &= -[\varphi_i, \phi_i] && (17)\\
\mathrm{q}_c a_{ij} &= b_{ij} - \phi_i + \lambda_{ij}(\phi_j) + \tfrac{1}{2}[a_{ij}, a_{ij}] && (18)\\
\mathrm{q}_c b_{ij} &= \sigma_i - \lambda_{ij}(\sigma_j) - [\varphi_i + \iota_{b_{ij}}, a_{ij}]\ . && (19)
\end{aligned}$$

Here $\mathrm{q}_c x$ for any field x is defined as $\mathrm{q}x + [c_i, x]$.

This BRST algebra closes on-shell [2]. By on-shell we mean that on double intersections $\mathcal{U}_{ij}$ the relationships

$$
\begin{aligned}
{}^{\lambda_{ij}*}m_j - m_i + \iota_{\gamma_{ij}} &= 0 \qquad (20)\\
{}^{\lambda_{ij}}c_j - c_i - \iota_{a_{ij}} &= 0 \qquad (21)\\
{}^{\lambda_{ij}}\pi_j - \pi_i + \iota_{\eta_{ij}} &= 0 \qquad (22)\\
{}^{\lambda_{ij}}\varphi_j - \varphi_i - \iota_{b_{ij}} &= 0\ , \qquad (23)
\end{aligned}
$$

are imposed. There are similar relationships on triple intersections, for a full discussion see [2]. Also, on-shell the BRST operator squares to the gauge transformation

$$
\mathrm{q}^2 x_i = [\varphi_i + \iota_{\phi_i}, x_i] \qquad (24)
$$

on any field x_i, except on η_{ij} and b_{ij}.

3.1 Isolated local charts

Let us set, temporarily, the intersection fields to trivial values

$$
\gamma_{ij} = \eta_{ij} = b_{ij} = a_{ij} = 0\ , \qquad (25)
$$

and work of a single chart $\mathcal{U}_i$. We can therefore omit the indices i from the formulae. If this is the case, the sum of the BRST transformations Q^+ and Q^- of Ref. [5, Eq. (2.24)] coincide with q with the following identifications:

$$
\begin{aligned}
A^{\mathrm{Twist}} &= m \qquad (26)\\
B^{\mathrm{Twist}} &= 2\ \iota^+_B \qquad (27)\\
C^{\mathrm{Twist}} &= 0 \qquad (28)\\
\psi^{\mathrm{Twist}} &= -\tfrac{1}{2}\pi \qquad (29)\\
\tilde{\psi}^{\mathrm{Twist}} &= \sqrt{2}\ \iota^+_\alpha \qquad (30)\\
\chi^{\mathrm{Twist}} &= -\sqrt{2}\ \iota^+_{\mathrm{d}_m E} \qquad (31)\\
\tilde{\chi}^{\mathrm{Twist}} &= -\tfrac{1}{2}\iota_E \qquad (32)\\
\phi^{\mathrm{Twist}} &= \tfrac{1}{2\sqrt{2}}\ \varphi \qquad (33)\\
\bar{\phi}^{\mathrm{Twist}} &= -\tfrac{1}{2\sqrt{2}}\ \iota_\phi \qquad (34)\\
\zeta^{\mathrm{Twist}} = \eta^{\mathrm{Twist}} &= -\tfrac{1}{4}\ \iota_\sigma \qquad (35)\\
\tilde{H}'^{\mathrm{Twist}} &= \tfrac{1}{2}\ \iota_\rho \qquad (36)\\
H'^{\mathrm{Twist}} &= -\sqrt{2}\ \iota^+_{\left(-\mathrm{d}_m\,\rho + [\nu,\phi] + [\pi,E]\right)}\ . \qquad (37)
\end{aligned}
$$

The BRST operator splits $\mathrm{q} = \mathrm{Q}^+ + \mathrm{Q}^-$, and we have (for $c_i = 0$)

$$\begin{aligned} \mathrm{Q}^+ m &= \pi & \mathrm{Q}^- m &= \iota_E \\ \mathrm{Q}^+ B &= \alpha & \mathrm{Q}^- B &= \mathrm{d}_m E \\ \mathrm{Q}^+ \pi &= \mathrm{d}_m \varphi & \mathrm{Q}^- \pi &= \iota_\rho \\ \mathrm{Q}^+ E &= -\rho & \mathrm{Q}^- E &= \mathrm{d}_m \phi \\ \mathrm{Q}^+ \alpha &= -[\varphi, B] & \mathrm{Q}^- \alpha &= \mathrm{d}_m \rho - [\nu, \phi] - [\pi, E] \\ \mathrm{Q}^+ \varphi &= 0 & \mathrm{Q}^- \varphi &= -\iota_\sigma \\ \mathrm{Q}^+ \phi &= \sigma & \mathrm{Q}^- \phi &= 0 \\ \mathrm{Q}^+ \rho &= [\varphi, E] & \mathrm{Q}^- \rho &= \mathrm{d}_m \sigma + [\pi, \phi] \\ \mathrm{Q}^+ \sigma &= 0 & \mathrm{Q}^- \sigma &= -[\varphi, \phi] \ . \end{aligned} \tag{38}$$

The BRST operator of the non-Abelian gerbe involves also the anti-self-dual part of the two-forms appearing above. This means that the BRST algebra, with these identifications, forms a self-consistent extension of the twisted algebra where only the self-dual part appears. Note, however, that for this comparison we had to set C to zero and $\eta = \zeta$ in the twisted theory.

The above restrictions mean that the sector we are interested in is not quite a balanced topological quantum field theory [21], because ghost number grading is different (*cf.* Table 1) and we have replaced two fields η, ζ that in the twisted theory have opposite ghost number with a single field σ. Therefore, the usual arguments for the absence of ghost number anomaly are not quite valid. Despite our removing these restrictions in Sec. 3.2, this departure from balanced TQFT will become even more pronounced, as certain non-local effects will have to be incorporated in the formalism.

3.2 Intersections of local charts

The gauge field m in the non-Abelian gerbe is not the only one-form at our disposal, but we have also $m_i - \iota_{\gamma_{ij}}$. (This is of course the same as ${}^{\lambda_{ij}*}m_j$.) For this to make sense we must work on a double intersection $\mathcal{U}_{ij}$, and turn on all other fields supported on double intersections as well, η_{ij}, a_{ij}, and b_{ij}.

The BRST algebra of the gerbe turns out to be too large as such, however, and we have to restrict $a_{ij} = 0$. As then also $\mathrm{q}a_{ij} = 0$, we have the conditions

$$a_{ij} = 0 \tag{39}$$

$$b_{ij} = \phi_i - \lambda_{ij}(\phi_j) \ . \tag{40}$$

This is the same restriction as what was necessary in [2] to map the nilpotent BRST operator on the universal gerbe to the non-nilpotent operator that implemented the infinitesimal symmetries of a non-Abelian gerbe of [18]. It was shown in [2] in particular that on-shell these two equations can be imposed as algebraic identities.

These identities lead now to two simplifications in the constraints:

$$
\begin{aligned}
{}^{\lambda_{ij}}c_j - c_i &= 0 && (41)\\
{}^{\lambda_{ij}}(\varphi_j + \iota_{\phi_j}) - (\varphi_i + \iota_{\phi_i}) &= 0\ . && (42)
\end{aligned}
$$

Then $\varphi_i + \iota_{\phi_i}$ is globally well-defined section of a vector bundle. Also,

$$
\mathrm{q}^2 x = [\varphi + \iota_\phi, x] \tag{43}
$$

for any field $x = B_i, \eta_{ij}$ *etc.*

With these restrictions on the gerbe, the BRST operator in Ref. [5, Eq. (2.24)] reduces precisely to the BRST operator of the fully decomposed gerbe. The precise identifications, that essentially generalise the above-presented, are as follows:

$$
\begin{aligned}
A^{\mathrm{Twist}} &= m_i - \iota_{\gamma_{ij}} && (44)\\
B^{\mathrm{Twist}} &= 2\ \iota^{+}_{B_i} && (45)\\
\psi^{\mathrm{Twist}} &= -\tfrac{1}{2}(\pi_i - \iota_{\eta_{ij}}) && (46)\\
\tilde{\psi}^{\mathrm{Twist}} &= \sqrt{2}\ \iota^{+}_{\alpha_i} && (47)\\
\chi^{\mathrm{Twist}} &= -\sqrt{2}\ \iota^{+}_{\mathrm{d}_{m_i} E_i} && (48)\\
\tilde{\chi}^{\mathrm{Twist}} &= -\tfrac{1}{2}\iota_{\lambda_{ij}(E_j)} && (49)\\
\phi^{\mathrm{Twist}} &= \tfrac{1}{2\sqrt{2}}\ \varphi_i && (50)\\
\bar{\phi}^{\mathrm{Twist}} &= -\tfrac{1}{2\sqrt{2}}\ \lambda_{ij}(\phi_j) && (51)\\
C^{\mathrm{Twist}} &= \tfrac{1}{4\sqrt{2}}\ (\lambda_{ij}(\phi_j) - \phi_i) && (52)\\
\zeta^{\mathrm{Twist}} &= -\tfrac{1}{4}\ \iota_{\sigma_i} && (53)\\
\eta^{\mathrm{Twist}} &= -\tfrac{1}{4}\ \iota_{\lambda_{ij}(\sigma_j)} && (54)\\
\tilde{H}'^{\mathrm{Twist}} &= \tfrac{1}{2}\ \iota_{\lambda_{ij}(\rho_j)} && (55)\\
H'^{\mathrm{Twist}} &= -\sqrt{2}\ \iota^{+}_{\left(-\mathrm{d}_{m_i}\rho_i + [\kappa(m_i),\phi_i] - [B_i,\lambda_{ij}(\phi_i)] + [\pi_i, E_i]\right)}\ . && (56)
\end{aligned}
$$

It does not seem to be possible to define the operators Q^+ and Q^- separately, as this would require making sense for instance of

$$
\begin{aligned}
\mathrm{Q}^+(\phi_i - \lambda_{ij}(\phi_j)) &\stackrel{?}{=} \sigma_i && (57)\\
\mathrm{Q}^-(\phi_i - \lambda_{ij}(\phi_j)) &\stackrel{?}{=} -\lambda_{ij}(\sigma_j)\ . && (58)
\end{aligned}
$$

Only the sum is $\mathrm{Q}^+ + \mathrm{Q}^- = \mathrm{q}$ is well-defined, and the topological quantum field theory is not balanced.

3.3 Global structure

On isolated local neighbourhoods $\mathcal{U}_i$ we have replicated in Sec. 3.1 the structure of a standard twisted Yang-Mills theory. The minor differences that remain were

– Some twisted Yang-Mills fields are constrained

$$C = 0 \tag{59}$$
$$\eta = \zeta \tag{60}$$

so that the global flavour symmetry $\mathrm{SU}(2)_F$ is broken;

– The non-Abelian gerbe keeps track also of anti-self-dual components; and

– The ghost number grading is compatible with $\mathrm{SU}(2)_F$ in the gerbe but not in Yang-Mills.

These restrictions are enough to break the balanced structure of the standard twisted theory, though. On intersections of these neighbourhoods $\mathcal{U}_{ij}$ the topological theory on the gerbe is even further away from being balanced, as the BRST operator does not split any more $\mathrm{q} \neq \mathrm{Q}^+ + \mathrm{Q}^-$. This means that though the theory might be locally nearly holomorphic on $\mathcal{U}_i$, its global structure is certainly put together by using non-holomorphic rules on double intersections $\mathcal{U}_{ij}$.

In the construction of Sec. 3.2 there was no restriction on the fields at all, in fact all three scalars were active

$$2\sqrt{2}\ \phi^{\mathrm{Twist}} = \varphi_i \tag{61}$$
$$2\sqrt{2}\ \bar{\phi}^{\mathrm{Twist}} = -\iota_{\lambda_{ij}(\phi_j)} \tag{62}$$
$$4\sqrt{2}\ C^{\mathrm{Twist}} = \iota_{\lambda_{ij}(\phi_j)-\phi_i}\ . \tag{63}$$

Similarly, their superpartners were unconstrained

$$-\,4\ \zeta^{\mathrm{Twist}} = \iota_{\sigma_i} \tag{64}$$
$$-4\ \eta^{\mathrm{Twist}} = \iota_{\lambda_{ij}(\sigma_j)}\ . \tag{65}$$

This construction reduces to the earlier construction, of course, when φ_i, ι_{ϕ_i}, and σ_i are separately covariant. This does not follow from the covariance of c_i and $\varphi_i + \iota_{\phi_i}$ observed in (41) – (42) alone. The flavour symmetry $\mathrm{SU}(2)_F$ is broken in this case not by ghost number assignments but rather by Čech-degree and the local structure of the $(\varphi_i, \phi_i, \sigma_i)$ system.

In a local quantum field theory one would usually expect to find one degree of freedom per Planck volume. In the present theory, however, where two local constructions overlap we seem to have an increase in the number degrees of freedom, in terms of the new fields C^{Twist} and $\zeta^{\mathrm{Twist}} - \eta^{\mathrm{Twist}}$. This does not need to change the structure of the Hilbert space radically, because the theory is after all a topological quantum theory whose Hilbert space is expected to be finite dimensional. One can think of this data either as a locally defined twist of the global configuration, or as new degrees of freedom. In the former case this data is kept fixed in the path integral, and characterise the global configuration. In the latter case these fields describe new degrees of freedom on the overlaps, and

$$
\begin{array}{|c|c|}
\hline
\text{Field} & \text{Superpartner} \\
\hline
\gamma_{ij} & \eta_{ij} \\
\lambda_{ij}(\phi_j) - \phi_i & \lambda_{ij}(\sigma_j) - \sigma_i \\
\lambda_{ij}(E_j) - E_i & \lambda_{ij}(\rho_j) - \rho_i \\
\delta_{ij} & \lambda_{ij}(\alpha_j) - \alpha_i \\
\hline
\end{array}
\tag{66}
$$

Table 3: *Degrees of freedom and their superpartners on* $\mathcal{U}_{ij}$.

should be integrated over in a path integral. All of the new degrees of freedom on the overlaps $\mathcal{U}_{ij}$ with their superpartners are summarised fully in Table 3.

If we wish indeed to interpret these discontinuities in the various fields in Table 3 as new degrees of freedom and integrate over them in a path integral, giving fields on an open cover $\mathcal{U}_i$, $\mathcal{U}_{ij}$, $\mathcal{U}_{ijk}$, and so on is clearly not the right way to organise this data. This is because at a single point in *e.g.* a double overlap we have simultaneously three different sets of fields — those defined on $\mathcal{U}_i|_j$, $\mathcal{U}_j|_i$, and $\mathcal{U}_{ij}$.

The additional structure that we need in order to understand the local distribution of degrees of freedom is in fact a compatible triangulation on X, where every simplex v of maximal dimension carries an index i corresponding to a local chart where it is included $v \subset \mathcal{U}_i$, each codimension one simplex s carries similarly an index ij corresponding to an overlap $s \subset \mathcal{U}_{ij}$, and so forth. A similar procedure leads to Gawedzki's topology on the loop space of X, and can be used to write down an explicit formula for the holonomy of an Abelian n-gerbe in [22]. Consequently, though a field may be defined over all $\mathcal{U}_{ij}$, it might be physical only on codimension one simplexes $s \subset \mathcal{U}_{ij}$ included in the triangulation we have chosen.

In this sense overlaps $\mathcal{U}_{ij}$ can be thought of as virtual domainwall defects in the ambient spacetime X. Of course, overlaps are open subsets of X and a domainwall defect is usually a closed submanifold embedded in X, so that the two structures are quite different. The point is that were there a domainwall embedded in X, the degrees of freedom on it should be labelled in terms of data defined on $\mathcal{U}_{ij}$. To develop these ideas fully, one should find out in what extent an eventual path integral formulation of the theory really depends on such a triangulation, and whether degrees of freedom on the above codimension one simplexes really imply the presence of a physical domainwall.

It is interesting to note nevertheless that at least the Bosonic new degrees of freedom on such a virtual domainwall seem to include degrees of freedom localised on a physical domainwall in four dimensions: a vector γ_{ij} and a scalar $\lambda_{ij}(\phi_j) - \phi_i$. The fields here are Bosonic components of a supermultiplets on a superspace where the BRST symmetry acts by odd translations. Untwisting these supermultiplets (with the other Fermionic data on the overlap) would unfortunately seem to require more detailed knowledge about the physical phase

space, equations of motion and gauge fixing in particular.[1]

The next question is the number of degrees of freedom on triple intersections $\mathcal{U}_{ijk}$. We have not introduced new fields explicitly on these overlaps, and the only object carrying three Čech indices is the class of the gerbe g_{ijk} which we keep fixed.

In the Abelian case it is easy to check whether fields defined on a double overlap, say $x^{\rm A}_{ij}$, can be accounted for *locally* in terms of differences $x^{\rm A}_j - x^{\rm A}_i$: the check is simply that $x^{\rm A}_{ij}$ should be closed under the Čech coboundary operator

$$(\partial x^{\rm A})_{ijk} \quad = \quad x^{\rm A}_{ij} + x^{\rm A}_{jk} + x^{\rm A}_{ki} \ . \tag{67}$$

In the non-Abelian case the situation is not quite so clear: fields on different charts cannot be compared directly, as they must be mapped first in the right frame using the transition functions λ_{ij}. Given this structure one can nevertheless define the covariant Čech coboundary operator

$$(\partial_\lambda x)_{ijk} \quad = \quad x_{ij} + {}^{\lambda_{ij}}x_{jk} + {}^{\lambda_{ij}\lambda_{jk}}x_{ki} \tag{68}$$

and use it to check what happens to a field that is clearly a difference of local fields, say $x_{ij} = {}^{\lambda_{ij}}x_j - x_i$. Suppose x_i is a differential form of positive rank. Then the result is its commutator with the class of the gerbe

$$(\partial_\lambda x)_{ijk} \quad = \quad [g_{ijk}, x_i] \ . \tag{69}$$

This is the consistent result, and indicates that there are no new degrees of freedom localised on $\mathcal{U}_{ijk}$. To spell this out more directly, note that changing charts over a fixed point in $\mathcal{U}_{ij}$ the differential form x_i as expressed in terms of x_j gets shifted

$$x_i \quad = \quad {}^{\lambda_{ij}}x_j - x_{ij} \ . \tag{70}$$

Repeating this procedure three times through $\mathcal{U}_{ij} \longrightarrow \mathcal{U}_{jk} \longrightarrow \mathcal{U}_{ki}$ we get

$$\begin{aligned} x_i \quad &= \quad {}^{\lambda_{ij}\lambda_{jk}\lambda_{ki}}x_i - (\partial_\lambda x)_{ijk} && (71)\\ &= \quad x_i \ . && (72) \end{aligned}$$

There is therefore no inconsistency in how x_{ij}, x_{jk}, and x_{ki} are defined, and no new degrees of freedom on $\mathcal{U}_{ijk}$.

These identifications put then the class g_{ijk} directly in evidence. On a triple intersection $\mathcal{U}_{ijk}$ we have three different twisted scalar fields, including $C^{\rm Twist}$. It is easy to see that the departure of the simplifications of the local construction on $\mathcal{U}_i$ gives rise to

$$4\sqrt{2}\Big(\partial_\lambda C^{\rm Twist}\Big)_{ijk} \quad = \quad \iota_{[g_{ijk},\phi_i]} \tag{73}$$

$$4\Big(\partial_\lambda(\zeta^{\rm Twist} - \eta^{\rm Twist})\Big)_{ijk} \quad = \quad \iota_{[g_{ijk},\sigma_i]} \ . \tag{74}$$

[1] Note that δ_{ij} should be seen as a part of the curvature of the global configuration, and that $\lambda_{ij}(E_j) - E_i$ can be absorbed in η_{ij}. Though the interpretation of these fields must be left open at this stage, their presence on the overlap may reflect the intricate structure of a non-Abelian gerbe rather than new degrees of freedom.

Field	Superpartner
$\tilde{\mathrm{d}}_m g_{ijk}$	$[\pi_i, g_{ijk}]$
$[g_{ijk}, \phi_i]$	$[g_{ijk}, \sigma_i]$
$[g_{ijk}, E_i]$	$[g_{ijk}, \rho_i]$
$[\nu_i, g_{ijk}]$	$[g_{ijk}, \alpha_i]$

(75)

Table 4: *Degrees of freedom and their superpartners on $\mathcal{U}_{ijk}$ from overlaps of fields defined on $\mathcal{U}_{ij}$. The table has been obtained by operating ∂_λ on Table 3.*

The same calculation for all new fields on double intersections is performed in Table 4. In all of these cases the Čech coboundary on $\mathcal{U}_{ijk}$ is merely the non-Abelian flux associated to an underlying field, and would not seem to indicate the presence of additional degrees of freedom.

To summarise, the local BRST operator of the twisted $N = 4$ theory on a local patch $\mathcal{U}_i$ does not involve *a priori* any of the cocycle data (g_{ijk}, λ_{ij}) in its definition. If the underlying structure is not well-defined as a principal bundle but rather as a non-Abelian gerbe, we need to consider the gauge theory on intersections of these local descriptions separately. Then the automorphisms λ_{ij} appear in the definition of twisted fields on the double intersections $\mathcal{U}_{ij}$, and the group-element g_{ijk} appears as a consequence of this as the "discrepancy" in the three different twisted theories on $\mathcal{U}_{ijk}$.

4 S-duality and self-duality

The global structure of the non-Abelian gerbe is much looser than that of a principal bundle. This allows us to make use of some of the full quantum structure of the $N = 4$ Yang-Mills theory in finding globally well-defined configurations. The idea is that local descriptions on different charts $\mathcal{U}_i$ and $\mathcal{U}_j$ may be related by a non-perturbative symmetry of the theory, such as S-duality. This category of solutions of the quantum theory is related to non-geometric backgrounds *cf.* [6].

Apart from describing a new category of twisted $N = 4$ Yang-Mills configurations, this will contribute in developing intuition of the physical significance of the fields that characterise a non-Abelian gerbe, namely the curvature triple that consists of

– The *curvature* $\omega_i \in \Omega^3(\mathcal{U}_i, \mathrm{Lie}\, G)$

$$\omega_i = \mathrm{d}_{m_i} B_i \ ; \tag{76}$$

– The *intermediate curvature* $\delta_{ij} \in \Omega^2(\mathcal{U}_{ij}, \mathrm{Lie}\, G)$

$$\delta_{ij} = \lambda_{ij}(B_j) - B_i + \mathrm{d}_{m_i}\, \gamma_{ij} - \frac{1}{2}[\gamma_{ij}, \gamma_{ij}] \ ; \tag{77}$$

– The *fake curvature* $\nu_i \in \Omega^2(\mathcal{U}_i, \mathrm{Lie\,Aut}\, G)$

$$\nu_i = \kappa(m_i) - \iota_{B_i} \ . \tag{78}$$

For properties of these differential forms, see [2].

The S-duality transformation S acts on the complex coupling and the field strength by

$$\tau \quad \longrightarrow \quad -\frac{1}{\tau} \tag{79}$$

$$\kappa(m) \quad \longrightarrow \quad \star\kappa(m) \ . \tag{80}$$

As we do not concern ourselves with the action principle here, it is only the latter that will be reflected in the structure of the non-Abelian gerbe. Since local connections on different charts are related only by the rather loose constraint

$${}^{\lambda_{ij}*}m_j - m_i + \iota_{\gamma_{ij}} \quad = \quad 0 \ , \tag{81}$$

we can construct a non-geometric configuration where

$${}^{\lambda_{ij}}\kappa(m_j) \quad = \quad \star\kappa(m_i) \tag{82}$$

without implying too restrictive assumptions. (Here $*$ denotes gauge transformation and $\star$ is the Hodge star.) When λ_{ij} is a trivial automorphism, this describes a non-geometric background where the field m_i on $\mathcal{U}_i$ is the electric gauge potential, and the field m_j on $\mathcal{U}_j$ is the magnetic gauge potential. They are directly related to each other only at the intersection $\mathcal{U}_{ij}$, where the difference is given by (81). The one-form γ_{ij} appears as an effective gauge field on the double intersection when the intersection is interpreted as a defect. As λ_{ij} acts on automorphisms by conjugation, traces remain invariant, and the two instanton number densities on $\mathcal{U}_{ij}$ coincide.

4.1 Consistency conditions

To see what constraint (82) does imply, we should expand it as

$$(1-\star)\kappa(m_i)|_j \quad = \quad \iota_{\mathrm{d}_{m_i}\gamma_{ij} - \frac{1}{2}[\gamma_{ij},\gamma_{ij}]} \ . \tag{83}$$

This means that on the intersection $\mathcal{U}_{ij}$ we must be able to write the fixed anti-self-dual two-form $(1-\star)\kappa(m_i)|_j$ as an exact (combinatorial) differential of a one-form γ_{ij} as in the above formula (83). Note that if $\kappa(m_i)$ happens to be purely self-dual, as is the case for the solutions of the standard twisted Yang-Mills theory, we are at liberty to choose the trivial solution $\gamma_{ij} = 0$.

Imposing an analogue of (82) on every double intersection $\mathcal{U}_{ij}$, $\mathcal{U}_{jk}$, and $\mathcal{U}_{ki}$ gives rise to a consistency condition on their respective intersection $\mathcal{U}_{ijk}$. There the situation depends on how m_k is related to m_i, m_j. If we indeed assume that the duality relation (82) holds in every case ij, jk, and ki, we find using (68) that the consistency condition (83) implies consistency on the triple intersection as well

$$\begin{aligned} \partial_\lambda\Big({}^{\lambda_{ij}}\kappa(m_j) - \star\kappa(m_i)\Big) &= \partial_\lambda\tilde{\delta}_{m_i}\gamma_{ij} - \tilde{\delta}_{m_i}\partial_\lambda\gamma_{ij} && (84)\\ &= 0 && (85)\end{aligned}$$

in the notation of [2]. Here ∂_λ is a λ-covariant Čech-differential (68); the check is that we can change charts $ij \longrightarrow jk \longrightarrow ki$ in such a way that we come back to where we started.

In this specific configuration on a Euclidean manifold ($\star^2 = 1$) the commutator of the field strength with the class of the gerbe g_{ijk} is anti-self-dual

$$(1-\star)\kappa(m_i)|_{jk} = [\kappa(m_i), \iota_{g_{ijk}}]\ . \tag{86}$$

This means that, on triple intersections, the self-dual part of every local field strength $\kappa(m_i)|_{jk}$ commutes with g_{ijk}. Hence, the class g_{ijk} determines a local Abelian system of self-dual fields in each $\mathcal{U}_{ijk}$. Suppose next that the two-form B_i vanishes everywhere. Then the curvatures of the non-Abelian gerbe summarise the construction

$$\begin{aligned} \omega_i &= 0 && (87)\\ \delta_{ij} &= \mathrm{d}_{m_i}\gamma_{ij} - \frac{1}{2}[\gamma_{ij},\gamma_{ij}] && (88)\\ \nu_i &= \kappa(m_i)\ . && (89)\end{aligned}$$

It is now not ν_i that is required to be self-dual as in twisted Yang-Mills, but rather δ_{ij}. The present structure is therefore characterised by the following constraints

$$\begin{aligned} B_i &= 0 && (90)\\ \iota_{\delta_{ij}} &= \star\iota_{\delta_{ij}} && (91)\\ {}^{\lambda_{ij}}\nu_j &= \star\nu_i\ . && (92)\end{aligned}$$

4.2 The self-dual gerbe

More generally, the above construction is an example of self-dual non-Abelian gerbes on four-manifolds satisfying

$$\begin{aligned} \delta_{ij} &= \star\delta_{ij} && (93)\\ {}^{\lambda_{ij}}\nu_j &= \star\nu_i\ . && (94)\end{aligned}$$

The relation between the curvature triple $(\omega_i, \delta_{ij}, \nu_i)$ and the cocycle that classifies the underlying gerbe topologically (g_{ijk}, λ_{ij}) is as follows:

– λ_{ij} is the action of Hodge duality on fake curvature on $\mathcal{U}_{ij}$; and

– g_{ijk} determines to what part of the Lie-algebra $\ker \iota_{g_{ijk}}$ the self-dual part of ν_i is restricted on $\mathcal{U}_{ijk}$.

These assumptions imply in particular $[\delta_{ij}, g_{ijk}] = 0$.

The effect of allowing B_i to be non-zero is to relax the anti-self-duality condition (82) somewhat, by subtracting an inner automorphism part ι_B from the respective field strengths that the condition relates. The curvature ω_i may now be non-zero, and measures precisely this departure from the initial self-duality condition (82).

[As an aside, an other conceivable route of embedding this non-geometric background in a gerbe would have been to set $\delta_{ij} = 0$, and parametrising the anti-self-dual part of $\kappa(m_i)$ by B_i

$$(1-\star)\kappa(m_i)|_j \quad = \quad \iota_{B_i - \lambda_{ij}(B_j)} \ . \tag{95}$$

Under this assumption, however, consistency requires the vanishing of

$$\partial_\lambda\Big({}^{\lambda_{ij}}\kappa(m_j) - \star\kappa(m_i)\Big) = [g_{ijk}, \nu_i] \tag{96}$$

on triple intersections. This means that g_{ijk} determines an Abelian frame for the restrictions of the *whole* fake curvature; such assumptions have the tendency of making the gerbe effectively Abelian.]

5 The flux of a non-Abelian gerbe

In this section we shall first discuss how 't Hooft's magnetic flux appears traditionally in Yang-Mills theory. This flux is classified in $H^2(X, \mathrm{Z}\,G)$, and can be thought of in terms of the change of the gauge group from electric to magnetic. A similar loosening of structure leads to magnetic flux associated to the class of a non-Abelian gerbe in $H^1(X, G \ltimes \mathrm{Aut}\, G)$.

5.1 't Hooft's Abelian magnetic fluxes

In $N = 4$ super-Yang-Mills all fields are in the adjoint representation, and the gauge group is $G/\,\mathrm{Z}\,G$ rather than the full exponential group of the Lie-algebra. We will consider in what follows the special unitary case of $G = \mathrm{SU}(n)/\mathbb{Z}_n$. The magnetic dual of this group is the full special unitary group $G^{\mathrm{v}} = \mathrm{SU}(n)$ with the centre restored [12]. (Another interesting example is the pair $G = \mathrm{Spin}(8)$, $G^{\mathrm{v}} = \mathrm{Spin}(8)/\mathbb{Z}_2 \times \mathbb{Z}_2$. Same observations apply.)

Consider an "electric" principal bundle E with transition functions h_{ij} valued in the gauge group $G = \mathrm{SU}(n)/\mathbb{Z}_n$. Choose a lift from $G = \mathrm{SU}(n)/\mathbb{Z}_n$ to $G^{\mathrm{v}} = \mathrm{SU}(n)$. On a triple intersection $\mathcal{U}_{ijk}$ the lifted transition functions $\hat{h}_{ij}$ do not necessarily satisfy the usual cocycle condition, but there may be an obstruction

$$\hat{h}_{ij}\hat{h}_{jk}\hat{h}_{ki} \quad = \quad a_{ijk} \ , \tag{97}$$

where $a_{ijk} \in \mathrm{Z}\,G$. If $G = \mathrm{SU}(n)$, we can think of these Abelian obstructions in terms of $n \times n$ matrixes

$$a_{ijk} \quad = \quad \mathrm{e}^{2\pi i \frac{k_{ijk}}{n}} \mathbf{1}_n \ , \qquad k_{ijk} \in \mathbb{Z} \ . \tag{98}$$

We may attempt to remove this obstruction by changing our choice of lift consistently on each intersection

$$\hat{h}'_{ij} \quad = \quad \hat{h}_{ij} k_{ij} \ , \qquad k_{ij} \in \mathrm{Z}\,G \ . \tag{99}$$

If it turns out that the mismatch a_{ijk} cannot be compensated for by changing the lift in this way, we have a true obstruction $[a_{ijk}] \in \check{H}^2(X, \mathbb{Z}_n)$ to the lift. On the other hand, if it turns out that $[a_{ijk}] = 0$, then the lifted bundle $\hat{E}$ exists as a globally well-defined entity.

One may look for such obstructions [10,11] by calculating Wilson loops along closed paths. The magnetic flux captured inside the loop is precisely the above obstruction $[k_{ijk}]$. (The exponential of this $[a_{ijk}]$ is rather the surface holonomy associated to this magnetic flux; they both describe the same physics.) In the electric picture we have therefore a well-defined G-bundle E. In the magnetic picture no such global G^{v}-bundle exists unless the (torsion class) magnetic flux $[a_{ijk}]$ vanishes. If it does not vanish, the global structure on the magnetic side is a flat Abelian gerbe, rather than a principal G^{v}-bundle. This magnetic flux satisfies the cocycle condition

$$a_{jkl} a_{ijl} = a_{ijk} a_{ikl} \ . \tag{100}$$

5.2 Outer automorphisms

Suppose we are given locally a well-defined principal G-bundle P_i on each local neighbourhood $\mathcal{U}_i$, and invertible mappings $\lambda_{ij} : P_j \longrightarrow P_i$ that act by automorphisms $\mathrm{Aut}\,G$ on the fibre G. The automorphisms need not be just conjugations by a group element, but could well be outer automorphisms, such as complex conjugation for $G = \mathrm{SU}(n)$, or triality for $G = \mathrm{Spin}(8)$.

Given a general automorphism λ_{ij}, there is no universal split to inner and outer automorphisms. As the latter are defined as the quotient $\mathrm{Aut}\,G/\,\mathrm{Int}\,G = \mathrm{Out}\,G$, we can nevertheless project an automorphism to its outer part $p(\lambda_{ij}) = w_{ij}$. Suppose we are given such a pure outer automorphism on each intersection $\mathcal{U}_{ij}$ that satisfies

$$w_{ij} w_{jk} w_{ki} \quad = \quad \mathbf{1} \ . \tag{101}$$

This amounts to choosing a class

$$[w] \in H^1(X, \mathrm{Out}\,G) \ , \tag{102}$$

and determines a principal $\mathrm{Out}\,G$-bundle in $\mathrm{Tors}\,\mathrm{Out}\,G$. The most obvious example of this structure is perhaps Yang-Mills on a local $\mathrm{Out}\,G$-orbifold. Then,

the surface holonomy of an Abelian gerbe picks up discrete torsion that can be understood in precisely these terms [23].

If we lift these outer automorphisms from $\mathrm{Out}\, G$ to $\lambda_{ij} \in \mathrm{Aut}\, G$, the consistency condition (101) is replaced by

$$\lambda_{ij}\lambda_{jk}\lambda_{ki} \quad = \quad \iota_{g_{ijk}} \tag{103}$$

for some mapping to the group $g_{ijk} : \mathcal{U}_{ijk} \longrightarrow G$. This will complicate the cocycle condition satisfied by g_{ijk}, however.

Consider the case $G = \mathrm{SU}(n)$, so that $\mathrm{Z}\, G$ are given by nth roots of unity. Suppose that $p(\lambda_{ij})$ acts by complex conjugation, and λ_{jk}, λ_{ki} are pure conjugations. Because the complex conjugation will act also on λ_{jk}, λ_{ki} in the definition of $\iota_{g_{ijk}}$, it is clear that a direct analogue of the Abelian cocycle condition (100) will not be satisfied. The appropriate generalisation will lead us to the topic of the next section:

5.3 Crossed modules

The problem of generalising (100) to an equation that could be valid also for non-Abelian cocycles can be solved, when a way to keep track of the "frame" in which a group element is given is developed. The right structure for this is the *crossed module*: This structure consists of the groups G and H, the homomorphism $\partial : G \longrightarrow H$ and the action of $h \in H$ on $g, g' \in G$ denoted *e.g.* by $g \mapsto {}^h g$. The homomorphism ∂ is required to satisfy

$$\partial({}^h g) \quad = \quad \iota_h(\partial g) \tag{104}$$

$${}^{\partial g}(g') \quad = \quad \iota_g(g') \; . \tag{105}$$

We shall be interested in the case $H = \mathrm{Aut}\, G$ when the homomorphism $\partial = \iota$ is the conjugation by a group element.

We will consider, in particular, the group-valued function $g_{ijk} \in G$ on $\mathcal{U}_{ijk}$ and the automorphism-valued function $\lambda_{ij} \in \mathrm{Aut}\, G$ on $\mathcal{U}_{ij}$. This pair (g_{ijk}, λ_{ij}) defines locally an element of the crossed module $G \ltimes \mathrm{Aut}\, G$. The cocycle equations [17] that they satisfy are

$$\lambda_{ij}(g_{jkl})g_{ijl} \quad = \quad g_{ijk}g_{ikl} \tag{106}$$

$$\iota_{g_{ijk}}\lambda_{ik} \quad = \quad \lambda_{ij}\lambda_{jk} \; . \tag{107}$$

Two equivalent cocycles (g_{ijk}, λ_{ij}) and $(g'_{ijk}, \lambda'_{ij})$ differ by a coboundary; the coboundary equations [17,18] are quite involved due to the fact that for writing them down one should decompose the gerbe fully. In fact, the data that goes in this decomposition is effectively the data that is included in the differential geometry of such a fully-decomposed gerbe.

When these equivalencies are taken in account correctly, such a cocycle pair (modulo the coboundary relations) determines a cohomology class of a non-Abelian gerbe

$$[(g_{ijk}, \lambda_{ij})] \quad \in \quad H^1(X, G \ltimes \mathrm{Aut}\, G) \; . \tag{108}$$

This group is the direct generalisation of the Čech-cohomology group $H^1(X, G)$ whose elements determine isomorphism classes of principal G-bundles, *i.e.* $\mathrm{Tors}\, G$. We shall denote $\mathbf{G} = G \ltimes \mathrm{Aut}\, G$. Sometimes also the notation $G \xrightarrow{\iota} \mathrm{Aut}\, G$ is used as it emphasises the rôle played by the homomorphism ι.

We have already encountered two examples of such a cocycle, namely the Abelian magnetic flux $(a_{ijk}, \mathbf{1})$, and the outer automorphisms $(\mathbf{1}, w_{ij})$. More generally, the cohomology group $H^1(X, \mathbf{G})$ of a non-Abelian gerbe fits in the exact sequence [24]

$$H^0(X, \mathrm{Out}\, G) \longrightarrow H^2(X, \mathrm{Z}\, G) \longrightarrow H^1(X, \mathbf{G}) \longrightarrow \mathrm{Tors}(\mathrm{Out}\, G) \ . \tag{109}$$

The image of elements $[g_{ijk}] \in H^2(X, \mathrm{Z}\, G)$ in $H^1(X, \mathbf{G})$ is $(g_{ijk}, \mathbf{1})$; the image of a general (g_{ijk}, λ_{ij}) in $\mathrm{Tors}(\mathrm{Out}\, G)$ is in the equivalence class of principal bundles given by $[p(\lambda)] \in H^1(X, \mathrm{Out}\, G)$.

A category of examples that carry this non-Abelian generalisation $[(g_{ijk}, \lambda_{ij})]$ of the more usual Abelian magnetic flux $[a_{ijk}]$ would be orbifold theories where the orbifold action λ_{ij} involves an arbitrary conjugation with a group element, and not just the outer part of the automorphism group. These theories are locally $N = 4$ supersymmetric outside the actual fixed point locus.

6 Chiral five-branes

I will include in this section a few remarks on eventual applications of the above observation on describing partially the worldvolume dynamics of a stack of chiral five-branes.

The uncompactified six-dimensional worldvolume theory for a single chiral M-theory five-brane involves the $N = (0, 2)$ tensor multiplet [25]. The tensor field couples to tensionless worldvolume strings whose dynamics give the parallel low-energy excitations of the worldvolume; the five scalar fields in the multiplet give the transverse excitations. At weak worldsheet coupling a stack of these branes has the worldvolume excitations of the Little String Theory [26].

Geometrically the two-form can be thought of as a connection on an Abelian gerbe on the worldvolume [27]. The Deligne class of the gerbe on the brane is twisted by the class of the bulk two-gerbe [9] in a direct analogue to what happens in String Theory [7]. This is also how the elusive E_8 structure [28] enters the geometry of gauge fields in M-theory [29]. It seems therefore reasonable that the low-energy dynamics of a stack of these branes should be described geometrically by a non-Abelian gerbe. The matter turns out to be much more subtle than that, owing *e.g.* to the inherent non-localities on the non-critical worldvolume string theory.

Reduced from six to four dimension, the tensor multiplet reduces however to the $N = 4$ vector multiplet. A reduction of an M-theory five-brane on a torus, in particular, can be related directly to the self-dual D3-brane [30] whose low-energy description is the $N = 4$ supersymmetric Yang-Mills theory. Wrapping the brane around a more general holomorphic cycle Σ breaks supersymmetry

by a further half, and one obtains the four-dimensional $N = 2$ super-Yang-Mills theory. The four-dimensional interpretation of the cycle Σ is that it is the Seiberg-Witten curve [31].

Consider a five-brane M that is locally of the form $\mathcal{U}_i \times \mathbb{T}^2$, where $\{\mathcal{U}_i\}$ is a cover of a Euclidean four-manifold X. On each $\mathcal{U}_{ij}$ the $\mathbb{T}^2$-fibres can be related one to an other by $\mathrm{SL}(2,\mathbb{Z})$ transformations that act precisely as S-duality transformations on the remaining degrees of freedom on X. As the worldvolume degrees of freedom are tensionless strings, there are massless winding modes in any limit we might consider. In the large $\mathrm{Vol}\,\mathbb{T}^2$ limit Kaluza-Klein modes are suppressed, however, and we get an (approximative) transverse $\mathrm{SO}(6)$ invariance in eleven dimensions.

The transverse $\mathrm{SO}(6)$ symmetry together with the fact that the five-brane breaks half of the supersymmetries in the bulk mean that the effective theory on X includes the $N = 4$ super-Yang-Mills theory. Even if this local quantum field theory misses some of the remaining massless non-local degrees of freedom on the five-brane, it is nevertheless a unitary quantum field theory, and we can consider it as a self-consistent sub-sector of the full worldvolume theory on M.

In a flat or hyper-Kähler background metric the twisted and the physical $N = 4$ super-Yang-Mills theory are equivalent [4, 16]. The observations in this paper can therefore be applied to five-branes on six-manifolds that are torus bundles over some hyper-Kähler manifold, such as K3 or an ALE space in the non-compact case. Abelian gerbes on toric fibrations and string compactifications on stacks have been discussed in the Abelian case in [32, 33]. For a discussion on Conformal Field Theory and branes, see [34].

A more direct relationship between the tensionless tensor theory and Yang-Mills could arise already in five dimensions; these twisted theories have not been worked out in detail, however. Indeed, a reduction of the five-brane theory on a circle yields the five-dimensional super-Yang-Mills theory with 16 supercharges. The spin-groups relevant to this theory are

$$\begin{array}{rclcrcl} \mathrm{Spin}^0(4,1) & = & \mathrm{Sp}_{1,1} & \subset & \mathrm{Spin}^0(5,1) & = & \mathrm{SL}(2,\mathbb{H}) \\ \mathrm{Spin}(5) & = & \mathrm{Sp}_2 & \subset & \mathrm{Spin}(6) & = & \mathrm{SU}(4) \end{array} . \tag{110}$$

In the Euclidean case the worldvolume spin group and the R-symmetry group are both Sp_2, and we can twist the theory by identifying the two. This will give rise to a Fermionic two-form, a vector, and a scalar (*e.g.* "α, π, σ") from gaugini, and a Bosonic vector, say b, from scalars. A vector (in the Abelian case) is dual to a two-form in five dimensions $\mathrm{d}\,b = *\,\mathrm{d}\,B$.

It is interesting to note that if we had at our disposal a determinant-like homomorphism $\det_{\mathbb{H}} : \mathrm{Sp}_{1,1} \longrightarrow \mathrm{SU}(2)$ [35], we could use it to twist the five-dimensional theory with the diagonal subgroup $\mathrm{SU}(2) \subset \mathrm{Sp}_2$ of the R-symmetry group such that the R-symmetry representation of the four supercharges splits $\mathbf{4} \longrightarrow \mathbf{2} \oplus \mathbf{2}$. As the scalars are in the antisymmetric $\mathbf{5}$ of Sp_2 this means that they decompose to $\mathbf{3} \oplus \mathbf{1} \oplus \mathbf{1}$. The four-dimensional interpretation of this matter content is a self-dual two-form and two scalars — the third scalar needed in the

four-dimensional theory arises from the reduction of the gauge field. However, from the five-dimensional point of view this decomposition is also that of a massive vector field: it might be interesting to look for a massive version of the above Hodge duality, and a relationship to a (massive) tensor field in five dimensions.

7 Discussion

The link between the BRST operator of a non-Abelian gerbe and twisted Yang-Mills theory allows a generalisation of Yang-Mills theory where local structures are related to each other in a looser fashion than in a standard principal bundle. Where the local structure of standard Yang-Mills theory is determined by a gauge equivalence class of the transition functions h_{ij}, the data needed in this generalised structure is an element of a crossed module (g_{ijk}, λ_{ij}) determining a class in $H^1(X, \mathbf{G})$.

As the class of the gerbe depends both on g_{ijk} and λ_{ij}, these quantities do not really have invariant meaning separately. If we have chosen a specific representative (g_{ijk}, λ_{ij}) of a class $[(g_{ijk}, \lambda_{ij})] \in H^1(X, \mathbf{G})$, we may nevertheless try to see what the physical origin of these two quantities is. As explained in the Paper, λ_{ij} can be thought of as a generalisation of the transition functions in Yang-Mills theory, and g_{ijk} can be thought of as a non-Abelian generalisation of magnetic flux. Such an Abelian magnetic flux a_{ijk} showed up in lifting the transition functions to the magnetic gauge group

$$\widehat{s_i s_j^{-1}} = \hat{h}_{ij} \tag{111}$$

and comparing them over a triple intersection

$$\partial(\widehat{s_i s_j^{-1}}) \quad = \quad a_{ijk} \ ; \tag{112}$$

similarly, if we have three independent differential forms ϕ_i, ϕ_j, and ϕ_k defined over the same point in $\mathcal{U}_{ijk}$ in a non-Abelian gerbe, the respective discontinuities on $\mathcal{U}_{ij}$, $\mathcal{U}_{jk}$, and $\mathcal{U}_{ki}$ satisfy

$$\partial_\lambda\Big(\lambda_{ij}(\phi_j)\phi_i^{-1}\Big) \quad = \quad [g_{ijk}, \phi_i] \ . \tag{113}$$

(We use the multiplicative notation of combinatorial differential geometry to emphasise the analogy.)

Magnetic flux in a standard Yang-Mills theory leads of course to a milder loosening of the electric structure of the theory. This flux can be classified in terms of centre valued classes in $H^2(X, \mathrm{Z}\, G)$. The invariant statement is that the class of the gerbe $[(g_{ijk}, \lambda_{ij})]$ generalises that Abelian magnetic flux $[a_{ijk}]$ to a non-Abelian context in the sense of the exact sequence (109).

The local structure of the thus loosened theory gives rise to new degrees of freedom localised on double intersections of local charts, where two conflicting

theories overlap. We have argued that it is useful to think of these overlaps as domainwalls with dynamics given by fields either switched off in the bulk theory (C^{Twist} and $\zeta^{\text{Twist}} - \eta^{\text{Twist}}$) or arising from the mismatch of the local fields in the two neighbourhoods (γ_{ij}). Technically this required choosing a triangulation of X compatible with the cover we use $\{\mathcal{U}_i\}$, and attaching an index i to each volume in the triangulation, ij codimension one simplex, ijk codimension two simplex, and so on. Then the domainwall degrees of freedom are indeed localised in a (network) of simplexes labelled by index pairs ij.

It was further argued that there are no new degrees of freedom in the codimension two simplexes $\Sigma_{ijk} \subset \mathcal{U}_{ijk}$ labelled by index triples ijk. These are generically codimension two surfaces, and correspond in four dimensions to (Euclidean) string worldsheets. The class of the gerbe involves nevertheless the fixed mappings $g_{ijk} : \Sigma_{ijk} \longrightarrow G$. As argued above, this map is a generalisation of the Abelian magnetic flux that arises as the centre part of a Wilson line in magnetic configurations. It plays therefore naturally the rôle of a surface holonomy of the non-Abelian gerbe over the surface Σ_{ijk}. This generalisation requires, of course, revising what usually is meant by a surface holonomy, and somewhat side-steps problems arising in more direct definitions of surface holonomies that depend on a choice of surface ordering *e.g.* [36, 37]. For generalisations that make use of non-Abelian two-forms see *e.g.* [38, 39].

These domainwalls can be identified in fact with membranes moving inside the four-dimensional bulk space. The worldvolume theory on them is indeed always the super-Yang-Mills theory reduced from ten dimensions, in this case *via* the twisted four-dimensional theory. Due to the topological nature of these membranes, one might suspect that they are related to the topological Dirac branes U_3 on the five-brane worldvolume whose boundaries are the tensionless worldvolume strings $W_2 = \partial U_3$ in the notation of Ref. [8]. This structure is in fact required in order to embed a stack of interacting membranes in the five-brane worldvolume.

Apart from the non-Abelian fluxes and the defect dynamics, an other new aspect in quantum field theory is how the non-local structure of the fields on the gerbe generalises the global structure of the twisted theory: Indeed, on a triple intersection we were consequently forced to consider three different scalar fields C^{Twist}, whose covariant difference — in the sense explained in (68) — was related to one of the local fields on the gerbe ϕ_i and the cocycle data g_{ijk}. This non-local structure came into its own when considering non-geometric backgrounds where gauge fields on adjacent charts were related by S-duality. It was possible to give an explicit formula for this relation consistently off-shell, as the relationship between the gauge fields was determined up to an arbitrary group-valued one-form γ_{ij}. It turned out that the rôle of the cocycle g_{ijk} in this case was to constrain the self-dual part on triple intersections.

As the ghost number assignments in the gerbe and in the twisted theory are different, the action principle will not be the same. In want of an action principle we have not been in a position to check that the above-mentioned non-geometric background reduces to the expected electric-magnetic dual background also on-

shell. This matter should clearly be clarified, as well as the construction of actions in general.

An other consequence of the difference in ghost number assignment is the fact that the topological Yang-Mills theory on the gerbe is not balanced, and that the partition function is therefore not protected from ghost number anomalies. This is interesting in view of constructing observables [2].

These observations have immediate implications for the study of the geometry of chiral five-branes. Though the $N = 4$ supersymmetric Yang-Mills theory captures only a part of the dynamics in those systems, the present generalisation allows the inclusion of some of the expected non-local phenomena in the field theory discussion, such as those related to Dirac membranes ending on tensionless strings, and the holonomies associated to the worldsheets of these strings.

Acknowledgements

I would like to thank Larry Breen, Chris Hull, and Bernard Julia for discussions, and Urs Schreiber and Eric Sharpe for correspondence. This research is supported by a Particle Physics and Astronomy Research Council (PPARC) Postdoctoral Fellowship.

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